

# DENOMINATOR-PRESERVING MAPS

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**ABSTRACT.** Let  $F$  be a continuous injective map from an open subset of  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Assume that, for infinitely many  $k \geq 1$ ,  $F$  induces a bijection between the rational points of denominator  $k$  in the domain and those in the image (the denominator of  $(a_1/b_1, \dots, a_n/b_n)$  being the l.c.m. of  $b_1, \dots, b_n$ ). Then  $F$  preserves the Lebesgue measure.

## 1. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

For every point  $u$  in  $\mathbb{Q}^n$  there exist uniquely determined relatively prime integers  $a_1, \dots, a_n, k$  such that  $k \geq 1$  and  $u = (a_1/k, \dots, a_n/k)$ . We then say that  $u$  is a *rational point* whose *denominator* is  $k$ , and write  $k = \text{den}(u)$ . A map  $F : U \rightarrow \mathbb{R}^n$ , where  $U$  is an open subset of  $\mathbb{R}^n$ , *preserves many denominators* (respectively, *all denominators*) if it induces a bijection between  $\{u \in U \cap \mathbb{Q}^n : \text{den}(u) = k\}$  and  $\{v \in F[U] \cap \mathbb{Q}^n : \text{den}(v) = k\}$  for infinitely many  $k$ 's (respectively, all  $k$ 's). In §6 we will give various examples of denominator-preserving maps. The map  $F$  *preserves the Lebesgue measure*  $\lambda$  if both  $F$  and  $F^{-1}$  send  $\lambda$ -measurable sets to  $\lambda$ -measurable sets, with preservation of the measure. We will prove the following theorem.

**Theorem 1.1.** *Let  $F : U \rightarrow \mathbb{R}^n$  be a continuous injective map that preserves many denominators. Then  $F$  preserves the Lebesgue measure.*

Let  $g(k)$  be the number of rational points of denominator  $k$  in the half-open cube  $(0, 1]^n$ . If  $f$  is a Riemann-integrable complex-valued function defined on a subset of  $\mathbb{R}^n$  then, by definition,  $f$  is bounded and has compact support (the latter being the closure of  $\{u : f(u) \neq 0\}$ ). We then consider  $f$  as defined on all of  $\mathbb{R}^n$ , by setting  $f(u) = 0$  for  $u \notin \text{dom}(f)$ . With this understanding, Theorem 1.1 is a consequence of the following fact.

**Theorem 1.2.** *For every Riemann-integrable function  $f$  we have*

$$\int_{\mathbb{R}^n} f d\bar{x} = \lim_{k \rightarrow \infty} \frac{1}{g(k)} \sum_{\text{den}(u)=k} f(u). \quad (*)$$

Theorem 1.2 has another corollary, which is interesting in its own right since it extends to an  $n$ -dimensional setting the old finding that the Farey enumeration of all rational numbers in  $[0, 1]$  is uniformly distributed (see [10], [7, p. 136], [6] and references therein).

**Theorem 1.3.** *Let  $u_1, u_2, u_3, \dots$  be an enumeration without repetitions of all rational points in the half-open cube  $(0, 1]^n$ . Assume that  $r \leq s$  implies  $\text{den}(u_r) \leq$*

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$\text{den}(u_s)$ . Then the sequence  $(u_r)$  is uniformly distributed. The same statement holds if  $(0, 1]^n$  is replaced by the closed cube  $[0, 1]^n$ .

In Theorem 1.1 we are not assuming any regularity for the map  $F$ . In §5 we will show that if  $F$  is differentiable at a point  $u$ , then the Jacobian matrix w.r.t. the standard basis is in  $\text{GL}_n \mathbb{Z}$  (this does not force  $F$  to be affine in a neighborhood of  $u$ ). As a corollary we get a version of Theorem 1.1 for bilipschitz maps, namely Corollary 5.4, whose proof does not depend on Theorem 1.2.

## 2. PROOF OF THEOREM 1.2

The function  $g(k)$  counting the number of points of denominator  $k$  in  $(0, 1]^n$  is Jordan's generalized totient [9, p. 11], which reduces to Euler's totient for  $n = 1$ .

**Lemma 2.1.** *Denoting the Möbius function by  $\mu$ , we have*

$$g(k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) d^n.$$

*Proof.* The number of rational points in  $(0, 1]^n$  that can be written in the form  $(a_1/k, \dots, a_n/k)$ , with  $a_1, \dots, a_n, k$  not necessarily relatively prime, is  $k^n$ . Since the set of such points is the disjoint union of the sets of points having denominator  $d|k$ , we also have  $k^n = \sum_{d|k} g(d)$ , and we apply the Möbius inversion formula.  $\square$

Given  $u = (t_1, \dots, t_n), v = (l_1, \dots, l_n) \in \mathbb{R}^n$ , we write  $u \leq v$  (respectively,  $u < v$ ) if  $t_i \leq l_i$  (respectively,  $t_i < l_i$ ) for every  $i$ . We denote the interval  $\{x \in \mathbb{R}^n : u < x \leq v\}$  by  $(u, v]$ , and we first establish the identity  $(*)$  in the statement of Theorem 1.2 for all characteristic functions  $f = \mathbb{1}_{(u, v]}$ . By partitioning  $(u, v]$  in finitely many subintervals of the form  $(u, v] \cap (w + (0, 1]^n)$ , with  $w \in \mathbb{Z}^n$ , we assume without loss of generality that  $u, v \in (0, 1]^n$ . Also, using the inclusion-exclusion principle it is easy to see that it suffices to establish  $(*)$  for intervals of the form  $(0, v]$ . Let then  $v = (l_1, \dots, l_n) \in (0, 1]^n$  and  $L = \prod l_i = \int \mathbb{1}_{(0, v]} d\bar{x}$ . Also, let  $h(k)$  be the number of points of denominator  $k$  in  $(0, v]$ ; we have to prove that  $\lim_{k \rightarrow \infty} h(k)/g(k) = L$ .

The number of rational points in  $(0, v]$  of the form  $(a_1/k, \dots, a_n/k)$ , with  $a_i, \dots, a_n, k$  not necessarily relatively prime, is  $\lfloor kl_1 \rfloor \cdots \lfloor kl_n \rfloor = \sum_{d|k} h(d)$ . By Möbius inversion we get

$$\begin{aligned} h(k) &= \sum_{d|k} \mu\left(\frac{k}{d}\right) \lfloor dl_1 \rfloor \cdots \lfloor dl_n \rfloor \\ &= \sum_{d|k} \mu\left(\frac{k}{d}\right) (dl_1 - \{dl_1\}) \cdots (dl_n - \{dl_n\}), \end{aligned}$$

where  $\{dl_i\} = dl_i - \lfloor dl_i \rfloor$  is the fractional part of  $dl_i$ . Now

$$(dl_1 - \{dl_1\}) \cdots (dl_n - \{dl_n\}) = d^n L + \sum_{\emptyset \neq J \subseteq \{1, \dots, n\}} (-1)^{|J|} d^{n-|J|} \prod_{\substack{j \in J \\ i \notin J}} l_i \{dl_j\},$$

and hence

$$h(k) = Lg(k) + \sum_{d|k} \mu\left(\frac{k}{d}\right) \left[ \sum_{\emptyset \neq J} (-1)^{|J|} d^{n-|J|} \prod_{\substack{j \in J \\ i \notin J}} l_i \{dl_j\} \right].$$

Dividing by  $g(k)$  and applying the triangle inequality we get

$$\begin{aligned} \left| \frac{h(k)}{g(k)} - L \right| &\leq \frac{1}{g(k)} \sum_{d|k} \left| \mu\left(\frac{k}{d}\right) \right| \left( \sum_{\emptyset \neq J} d^{n-|J|} \right) \\ &= \frac{1}{g(k)} \sum_{d|k} \left| \mu\left(\frac{k}{d}\right) \right| \left( \sum_{t=1}^n \binom{n}{t} d^{n-t} \right) \leq \frac{M}{g(k)} \sum_{d|k} \left| \mu\left(\frac{k}{d}\right) \right| d^{n-1}, \end{aligned}$$

for some  $M > 0$ , depending on  $n$  only.

Hence it suffices to show that

$$m(k) = \frac{\sum_{d|k} \left| \mu\left(\frac{k}{d}\right) \right| d^{n-1}}{\sum_{d|k} \mu\left(\frac{k}{d}\right) d^n}$$

tends to 0 as  $k$  tends to infinity. Since  $m$  is multiplicative, we can check this for  $k$  assuming prime-power values [5, Theorem 316]. We have

$$\begin{aligned} m(p^e) &= \frac{(p^e)^{n-1} + (p^{e-1})^{n-1}}{(p^e)^n - (p^{e-1})^n} \\ &= \frac{p^{en-e} + p^{en+1-e-n}}{p^{en} - p^{en-n}} \cdot \frac{p^{n+e-en}}{p^{n+e-en}} \\ &= \frac{p^n + p}{p^{n+e} - p^e} \\ &= \frac{p(p^{n-1} + 1)}{(p-1)(p^{n-1} + p^{n-2} + \dots + 1)} \cdot \frac{1}{p^e} \\ &\leq 4 \cdot \frac{1}{p^e}, \end{aligned}$$

that tends to 0 as  $p^e$  tends to infinity.

This establishes (\*) for the characteristic functions of intervals, and it is clear that (\*) must then hold for all finite linear combinations of such characteristic functions. The extension of (\*) to all Riemann-integrable functions is now straightforward (see, e.g., the proof of [7, Theorem 1.1.1]).

### 3. PROOF OF THEOREM 1.1

Let  $F, U$  be as in the statement of Theorem 1.1. As a consequence of Brouwer's invariance of domain theorem [8, p. 217],  $V = F[U]$  is open in  $\mathbb{R}^n$  and  $F : U \rightarrow V$  is a homeomorphism. By our assumptions on  $F$ , there exists a sequence  $k_1 < k_2 < k_3 < \dots$  such that  $F$  induces a bijection between the points of denominator  $k_i$  in  $U$  and those in  $V$ , for every  $i$ . By Theorem 1.2, for every Riemann-integrable function  $f$  whose support is contained in  $V$  we have

$$\begin{aligned} \int_{\mathbb{R}^n} f d\bar{x} &= \lim_{i \rightarrow \infty} \frac{1}{g(k_i)} \sum_{\substack{v \in V \\ \text{den}(v) = k_i}} f(v) \\ &= \lim_{i \rightarrow \infty} \frac{1}{g(k_i)} \sum_{\substack{u \in U \\ \text{den}(u) = k_i}} (f \circ F)(u) = \int_{\mathbb{R}^n} f \circ F d\bar{x}. \end{aligned}$$

Let  $W$  be an open subset of  $V$ . By the construction of the Lebesgue measure  $\lambda$  [14, proof of Theorem 2.14],  $\lambda(W)$  is the least upper bound of the values  $\int f d\bar{x}$ , where  $f$  ranges in the set of all continuous,  $[0, 1]$ -valued functions supported in  $W$ . Since this set of functions is in 1-1 correspondence —via postcomposition with  $F$ — with the analogous set of functions supported in  $F^{-1}W$ , the identity displayed above yields  $\lambda(F^{-1}W) = \lambda(W)$ . By [14, Theorem 2.14(c)],  $\lambda(F^{-1}A) = \lambda(A)$  for every  $A \subseteq V$  such that both  $A$  and  $F^{-1}A$  are  $\lambda$ -measurable, in particular for every Borel set. By [14, Theorem 2.17],  $A \subseteq \mathbb{R}^n$  is  $\lambda$ -measurable iff there exist  $B \in F_\sigma$ ,  $C \in G_\delta$  with  $B \subseteq A \subseteq C$  and  $\lambda(C \setminus B) = 0$ ; if this happens,  $\lambda(A) = \lambda(C)$ . Let then  $A \subseteq V$  be  $\lambda$ -measurable,  $B$  and  $C$  be as above with  $C \subseteq V$ . Then  $F^{-1}B \subseteq F^{-1}A \subseteq F^{-1}C$ , with  $F^{-1}B \in F_\sigma$  and  $F^{-1}C \in G_\delta$ . Since  $C \setminus B$  is Borel, we have by the above  $\lambda(F^{-1}C \setminus F^{-1}B) = \lambda(F^{-1}(C \setminus B)) = \lambda(C \setminus B) = 0$ ; hence  $F^{-1}A$  is  $\lambda$ -measurable and  $\lambda(F^{-1}A) = \lambda(F^{-1}C) = \lambda(C) = \lambda(A)$ . The same argument applies to  $F^{-1}$ , so the  $\lambda$ -measurability is preserved in both directions.

#### 4. PROOF OF THEOREM 1.3

Let  $X$  denote either the half-open cube  $(0, 1]^n$  of the closed cube  $[0, 1]^n$ . We recall that a sequence  $(u_r)_{r \in \mathbb{N}}$  in  $X$  is *uniformly distributed* if for every Riemann-integrable function  $f : X \rightarrow \mathbb{C}$  we have

$$\lim_{s \rightarrow \infty} \frac{1}{s} \sum_{r=1}^s f(u_r) = \int_X f d\bar{x}.$$

We let  $G(k)$  denote the number of rational points of denominator  $k$  in  $[0, 1]^n$ ; as in Lemma 2.1 we have

$$G(k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) (d+1)^n.$$

We need a few facts about the growth rate of  $g(k)$  and  $G(k)$ ; the following estimates are known [9, Exercises 1.3.4, 1.3.5, 1.5.3]:

- (a) for  $n = 1$ , there exist positive constants  $C_1, C_2$  such that

$$C_1 \frac{k}{1 + \log k} \leq g(k) \leq C_2 k;$$

- (b) for  $n \geq 2$ , there exist positive constants  $C_1, C_2$  (depending on  $n$ ) such that

$$C_1 k^n \leq g(k) \leq C_2 k^n.$$

Let  $t(k) = g(1) + g(2) + \cdots + g(k)$  and  $T(k) = G(1) + G(2) + \cdots + G(k)$ .

**Lemma 4.1.** *We have*

$$\lim_{k \rightarrow \infty} \frac{g(k+1)}{t(k)} = 0,$$

and analogously for  $G, T$ . Also,  $\lim_{k \rightarrow \infty} g(k)/G(k) = 1$ .

*Proof.* It is known [1, p. 155] that

$$1^n + 2^n + \cdots + k^n = \frac{1}{n+1} k^{n+1} + O(k^n).$$

The claim for  $g$  follows then easily from (a) and (b) above. As we now need specify the dimension  $n$ , we will write  $g_n$  for  $g$ , and analogously for  $G, t, T$ , till the end of the proof. Since the vertex  $(0, 0, \dots, 0)$  is contained in  $n$  maximal faces of the unit

cube  $[0, 1]^n$ , we have  $G_n(k) \leq g_n(k) + nG_{n-1}(k)$  (we set  $G_0$  to be identically 1). Therefore

$$\frac{G_n(k+1)}{T_n(k)} \leq \frac{g_n(k+1) + nG_{n-1}(k+1)}{t_n(k)},$$

and the latter tends to 0 as  $k$  tends to infinity.

For the last statement we observe that  $G_n(k) \leq g_n(k) + nG_{n-1}(k)$  implies

$$1 - n \frac{G_{n-1}(k)}{G_n(k)} \leq \frac{g_n(k)}{G_n(k)} < 1.$$

Also, we have the trivial bound

$$\frac{G_{n-1}(k)}{G_n(k)} \leq \frac{(k+1)^{n-1}}{g_n(k)},$$

whence  $\lim_{k \rightarrow \infty} G_{n-1}(k)/G_n(k) = 0$  by (a) and (b).  $\square$

Let  $(\alpha_r)_{r \in \mathbb{N}}$  be any element of  $\ell_\infty(\mathbb{C})$ , and let  $\beta \in \mathbb{C}$ . Three summation methods are relevant for us:

(1) *block convergence*

$$\lim_{k \rightarrow \infty} \frac{1}{g(k)} \sum_{r=t(k-1)+1}^{t(k)} \alpha_r = \beta;$$

(2) *Cesàro convergence*

$$\lim_{s \rightarrow \infty} \frac{1}{s} \sum_{r=1}^s \alpha_r = \beta;$$

(3) *blockwise Cesàro convergence*

$$\lim_{k \rightarrow \infty} \frac{1}{t(k)} \sum_{r=1}^{t(k)} \alpha_r = \beta.$$

Clearly (2)  $\Rightarrow$  (3), and (1)  $\Rightarrow$  (3) is an easy consequence of a theorem by Cauchy [7, Lemma 2.4.1]. Since, as proved in Lemma 4.1, the ratio  $g(k+1)/t(k)$  tends to 0, we have (3)  $\Rightarrow$  (2) by [7, Lemma 2.4.1]. Returning to the proof of Theorem 1.3, assume first  $X = (0, 1]^n$  and let  $(u_r)$  be a sequence as in the statement of Theorem 1.3. Fix a Riemann-integrable function  $f$  on  $X$ , let  $\alpha_r = f(u_r)$  and  $\beta = \int_X f d\bar{x}$ . The points  $\{u_r : t(k-1) < r \leq t(k)\}$  are precisely the points of denominator  $k$  in  $X$ , in some order, so (1) holds by Theorem 1.2. Therefore (2) holds as well and Theorem 1.3 is proved for the half-open cube.

Take now  $X = [0, 1]^n$ , and let  $(u_r)$ ,  $f$ ,  $(\alpha_r)$ ,  $\beta$  be defined as above with the obvious modifications. By Lemma 4.1  $G(k+1)/T(k) \rightarrow 0$ , so the above discussion holds verbatim and we only need to prove

$$\lim_{k \rightarrow \infty} \frac{1}{G(k)} \sum_{r=T(k-1)+1}^{T(k)} \alpha_r = \beta.$$

Without loss of generality the first  $g(k)$  points in  $u_{T(k-1)+1}, \dots, u_{T(k)}$  are in  $(0, 1]^n$ , and the remaining ones in  $[0, 1]^n \setminus (0, 1]^n$ . We thus get

$$\frac{1}{G(k)} \sum_{r=T(k-1)+1}^{T(k)} \alpha_r = \frac{g(k)}{G(k)} \cdot \frac{1}{g(k)} \sum_{r=T(k-1)+1}^{T(k-1)+g(k)} \alpha_r + \frac{1}{G(k)} \sum_{r=T(k-1)+g(k)+1}^{T(k)} \alpha_r.$$

For  $k$  tending to infinity  $g(k)/G(k)$  tends to 1 by Lemma 4.1, hence the first summand to the right-hand side tends to  $\beta$  as proved above. On the other hand the second summand tends to 0, because its absolute value is dominated by  $\|f\|_\infty(G(k) - g(k))/G(k)$ . This concludes the proof of Theorem 1.3.

**Remark 4.2.** In general block convergence is strictly stronger than blockwise Cesàro convergence; examples are easily constructed. It is precisely the stronger property (1) allowing us to ask for preservation of many denominators — rather than all denominators — in Theorem 1.1. This turns out to be handy, e.g., in Lemma 5.1 below. Equidistribution results for rational points in algebraic varieties are usually formulated in terms of blockwise Cesàro convergence. Points are sorted according to their height, and the resulting measure is the finite Tamagawa measure; see the survey [13].

## 5. TRANSLATIONS AND DIFFERENTIABILITY

Translations by rational vectors preserve many denominators.

**Lemma 5.1.** *Let  $v = d^{-1}(a_1, \dots, a_n) \in \mathbb{Q}^n$  have denominator  $d$ , let  $U \subseteq \mathbb{R}^n$  be open and let  $T_v : U \rightarrow \mathbb{R}^n$  be the translation by  $v$ . Then:*

- (i)  $T_v$  preserves every denominator  $k$  such that  $d \operatorname{rad}(d) | k$  ( $\operatorname{rad}(d)$  being the product of all prime factors of  $d$ , each taken with exponent 1);
- (ii)  $T_v$  preserves all denominators iff  $v \in \mathbb{Z}^n$ .

*Proof.* Denote by  $\mathfrak{o}_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{+\infty\}$  the  $p$ -adic valuation (i.e.,  $\mathfrak{o}_p(l)$  is the exponent at which the prime  $p$  appears in the unique factorization of  $l \in \mathbb{Q} \setminus \{0\}$ ). Let  $u = k^{-1}(b_1, \dots, b_n) \in U$  have denominator  $k$ , the latter being a multiple of  $d \operatorname{rad}(d)$ . We will show that  $k | \operatorname{den}(T_v(u))$ . Since  $\operatorname{den}(v) = \operatorname{den}(-v)$ , it will follow that both  $T_v$  and  $T_v^{-1}$  map points whose denominator is a multiple  $k$  of  $d \operatorname{rad}(d)$  to points whose denominator is a multiple of  $k$ , so that both  $T_v$  and  $T_v^{-1}$  must preserve such denominators. Let then  $p | k$ , with the intent of proving  $\mathfrak{o}_p(k) \leq \mathfrak{o}_p(\operatorname{den}(T_v(u)))$ .

For at least one index  $i$  we have  $p \nmid b_i$ , and for that index  $\mathfrak{o}_p(b_i/k) = \mathfrak{o}_p(1/k) < \mathfrak{o}_p(1/d) \leq \mathfrak{o}_p(a_i/d)$ ; the middle inequality is clear if  $p \nmid d$ , and follows from  $d \operatorname{rad}(d) | k$  otherwise. Therefore  $\mathfrak{o}_p(a_i/d + b_i/k) = \min\{\mathfrak{o}_p(a_i/d), \mathfrak{o}_p(b_i/k)\} = -\mathfrak{o}_p(k)$ , and hence  $0 < \mathfrak{o}_p(k) = -\mathfrak{o}_p(a_i/d + b_i/k) \leq \mathfrak{o}_p(\operatorname{den}(T_v(u)))$ . We thus proved (i); for the nontrivial direction of (ii), assume  $v \notin \mathbb{Z}^n$ . Then some prime  $p$  divides  $\operatorname{den}(v)$ , and the open set  $U$  surely contains a point  $u$  with  $p \nmid \operatorname{den}(u)$ . Therefore  $p | \operatorname{den}(T_v(u))$ , and  $T_v$  does not preserve all denominators.  $\square$

For the remaining of this paper we will be concerned only with maps that preserve all denominators. Under differentiability assumptions denominator-preserving maps have a rigid structure, as explained in the following theorem.

**Theorem 5.2.** *Let  $F : U \rightarrow \mathbb{R}^n$  be a continuous injective map that preserves all denominators. Assume that  $F$  is differentiable at  $u$  with Jacobian matrix  $A$  w.r.t. the standard basis of  $\mathbb{R}^n$ . Then  $A \in \operatorname{GL}_n \mathbb{Z}$ . In particular, if  $F$  is continuously differentiable on the open connected region  $V \subseteq U$ , then  $F$  has the form  $F(v) = Av + w$  on  $V$ , for some fixed  $A \in \operatorname{GL}_n \mathbb{Z}$  and  $w \in \mathbb{Z}^n$ .*

In §6 we will give an example of a denominator-preserving homeomorphism of  $\mathbb{R}^2$  that is differentiable at the origin, but fails to be affine in any neighborhood of

the origin. We will also construct a nowhere differentiable denominator-preserving homeomorphism of  $\mathbb{R}^2$ .

In order to prove Theorem 5.2 we need a few preliminaries. Let  $\sigma = (v_0, \dots, v_n)$  denote an ordered  $(n+1)$ -tuple of vectors in general position in  $U$ . We say that a sequence  $\sigma^1, \sigma^2, \sigma^3, \dots$  of such tuples, with  $\sigma^t = (v_0^t, \dots, v_n^t)$ , converges to the point  $u$  if  $u$  is in the convex hull of every  $\sigma^t$  and  $v_i^t$  converges to  $u$  for every  $i$ . Every  $\sigma$  determines a linear map  $L_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $L_\sigma(v_i - v_j) = F(v_i) - F(v_j)$ . The following calculus lemma might be known, but we have not been able to find a reference.

**Lemma 5.3.** *Let  $F$  be any map from an open subset  $U$  of  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Assume that  $F$  is differentiable at  $u \in U$  with differential  $L \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ , and let  $(\sigma^t)$  be a sequence converging to  $u$  as above, with  $v_i^t \in U$  for every  $t$  and  $i$ . Then  $L_{\sigma^t}$  converges to  $L$  in the operator norm.*

*Proof.* Fix temporarily  $t$ , let  $\sigma = \sigma^t = (v_0, \dots, v_n)$  and  $u = \sum_i \alpha_i v_i$ , with  $\sum_i \alpha_i = 1$  and  $\alpha_0, \dots, \alpha_n \geq 0$ . Denote by  $\sigma_i$  the tuple obtained from  $\sigma$  by replacing  $v_i$  with  $u$ . We claim that  $L_\sigma = \sum_i \alpha_i L_{\sigma_i}$ . Note that if  $u$  belongs to the affine subspace spanned by  $v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ , then  $L_{\sigma_i}$  is undefined, but this happens precisely when  $\alpha_i = 0$ , so the above identity still makes sense. Note also that if  $\beta_0, \dots, \beta_n \in \mathbb{R}$  are such that  $\sum_i \beta_i = 0$ , then  $L_\sigma(\sum_i \beta_i v_i) = \sum_i \beta_i F(v_i)$ ; this is easily proved by induction on the number of indices  $i$  such that  $\beta_i \neq 0$ .

Since both sides of the claimed equality are linear maps, it suffices to show that they agree on all differences  $v_j - v_k$ , i.e., that  $F(v_j) - F(v_k) = \sum_{\alpha_i \neq 0} L_{\sigma_i}(\alpha_i(v_j - v_k))$  for every  $j \neq k$ . We fix then  $j \neq k$ , and compute  $L_{\sigma_i}(\alpha_i(v_j - v_k))$  under the assumption  $\alpha_i \neq 0$ . We obtain:

- (i) if  $i \neq j, k$ , then  $L_{\sigma_i}(\alpha_i(v_j - v_k)) = \alpha_i(F(v_j) - F(v_k))$ ;
- (ii) if  $i = j$ , then  $L_{\sigma_j}(\alpha_j(v_j - v_k)) = F(u) - (\alpha_k + \alpha_j)F(v_k) - \sum_{l \neq j, k} \alpha_l F(v_l)$ ;
- (iii) if  $i = k$ , then  $L_{\sigma_k}(\alpha_k(v_j - v_k)) = -F(u) + (\alpha_j + \alpha_k)F(v_j) + \sum_{l \neq j, k} \alpha_l F(v_l)$ .

Indeed, (ii) is true since  $u = \sum_l \alpha_l v_l$  implies  $\alpha_j(v_j - v_k) = u - (\alpha_k + \alpha_j)v_k - \sum_{l \neq j, k} \alpha_l v_l$  and  $1 - (\alpha_k + \alpha_j) - \sum_{l \neq j, k} \alpha_l = 0$ , and (iii) is analogous. Summing up everything we obtain

$$\begin{aligned} \sum_{\alpha_i \neq 0} L_{\sigma_i}(\alpha_i(v_j - v_k)) &= (\alpha_j + \alpha_k)F(v_j) - (\alpha_k + \alpha_j)F(v_k) \\ &\quad + \sum_{i \neq j, k} \alpha_i(F(v_j) - F(v_k)) = F(v_j) - F(v_k), \end{aligned}$$

which settles our claim.

Let now  $\varepsilon > 0$ . Since  $F$  is differentiable at  $u$ , there exists an index  $t'$  such that for every  $i = 0, \dots, n$  and every  $t \geq t'$  the linear map  $L_{\sigma_i^t}$  is either undefined, or is in the open ball of center  $L$  and radius  $\varepsilon$  in  $\text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ . Since such a ball is convex, it contains  $L_{\sigma^t}$  by our claim above.  $\square$

We can now prove Theorem 5.2. Using the Mönkemeyer-Selmer multidimensional continued fractions algorithm (or any other topologically convergent procedure, see [12] for the Mönkemeyer-Selmer algorithm or [15] for a complete panorama) it is easy to construct a sequence  $(\sigma^t)$  converging to  $u$  such that each tuple  $\sigma^t = (v_0^t, \dots, v_n^t)$  is *unimodular*. By this we mean that every  $v_i^t$  is in  $\mathbb{Q}^n$  and the  $(n+1)$ -tuple  $(v_0^t, \dots, v_n^t)$  is unimodular.

1)  $\times (n + 1)$  integer matrix

$$S_t = \begin{pmatrix} a_{1,0}^t & \cdots & a_{1,n}^t \\ \vdots & \cdots & \vdots \\ a_{n,0}^t & \cdots & a_{n,n}^t \\ d_0^t & \cdots & d_n^t \end{pmatrix}$$

whose columns are the projective coordinates of  $v_0^t, \dots, v_n^t$ , with  $d_i^t = \text{den}(v_i^t)$ , is in  $\text{GL}_{n+1} \mathbb{Z}$ . Let analogously

$$W_t = \begin{pmatrix} b_{1,0}^t & \cdots & b_{1,n}^t \\ \vdots & \cdots & \vdots \\ b_{n,0}^t & \cdots & b_{n,n}^t \\ d_0^t & \cdots & d_n^t \end{pmatrix}$$

be the integer matrix whose columns are the projective coordinates of  $F(v_0^t), \dots, F(v_n^t)$ . Then the  $n \times n$  matrix  $A_t$  that expresses  $L_{\sigma^t}$  w.r.t. the standard basis of  $\mathbb{R}^n$  is the upper left minor of the  $(n + 1) \times (n + 1)$  matrix  $B_t$  defined by the identity

$$\begin{aligned} B_t & \begin{pmatrix} v_0 - v_n & \cdots & v_{n-1} - v_n & v_n \\ 0 & \cdots & 0 & 1 \end{pmatrix} \\ & = \begin{pmatrix} F(v_0) - F(v_n) & \cdots & F(v_{n-1}) - F(v_n) & F(v_n) \\ 0 & \cdots & 0 & 1 \end{pmatrix}; \end{aligned}$$

here the vectors on the top rows are  $n$ -rows column vectors. Multiplying to the right both sides of the above identity first by an appropriate elementary matrix, and then by the diagonal matrix whose diagonal entries are  $d_0^t, \dots, d_n^t$ , we get the identity  $B_t S_t = W_t$ , which implies that  $B_t = W_t S_t^{-1}$  has integer entries. Therefore  $A_t$  has integer entries. By Lemma 5.3  $A_t$  converges to  $A$  for  $t$  going to infinity, and since  $\text{Mat}_{n \times n} \mathbb{Z}$  is discrete in  $\text{Mat}_{n \times n} \mathbb{R}$  we conclude that  $A$  has integer entries. By Theorem 1.1  $F$  preserves the Lebesgue measure, so  $A$  is in  $\text{GL}_n \mathbb{Z}$ .

If  $F$  is continuously differentiable on the open connected region  $V$ , then clearly the Jacobian matrix  $A$  must be constant, since  $\text{GL}_n \mathbb{Z}$  is discrete. The map  $F(v) - Av$  has then null differential on  $V$ , so it is constant [3, (8.6.1)]; therefore  $F(v) = Av + w$  on  $V$ , for some fixed column vector  $w \in \mathbb{R}^n$ . As the translation by  $A^{-1}w$  equals the composite map  $A^{-1} \circ F : V \rightarrow \mathbb{R}^n$ , it preserves all denominators. By Lemma 5.1(ii)  $A^{-1}w \in \mathbb{Z}^n$ , and hence  $w \in \mathbb{Z}^n$ ; this concludes the proof of Theorem 5.2.

Theorem 5.2 yields a weaker version of Theorem 1.1, whose proof is independent of Theorem 1.2.

**Corollary 5.4.** *Let  $F$  be a bilipschitz homeomorphism of open subsets of  $\mathbb{R}^n$  that preserves all denominators. Then  $F$  preserves the Lebesgue measure.*

*Proof.* By Rademacher's theorem [4, Theorem 2, p. 81] both  $F$  and  $F^{-1}$  are differentiable  $\lambda$ -a.e.. Moreover, by [14, Lemma 7.25] the  $F^{-1}$ -image of the set of nondifferentiability points of  $F^{-1}$  is a Lebesgue nullset. Therefore, for  $\lambda$ -a.e.  $u \in \text{dom}(F)$ ,  $F$  is differentiable at  $u$  and  $F^{-1}$  is differentiable at  $F(u)$ . Let  $A$  be the Jacobian matrix of  $F$  at  $u$  and  $B$  that of  $F^{-1}$  at  $F(u)$ . By the proof of Theorem 5.2 both  $A$  and  $B$  have integer entries; since their product is the identity matrix, both of them are in  $\text{GL}_n \mathbb{Z}$ . Note that in the proof of Theorem 5.2 we employed Theorem 1.2



only in showing  $|\det(A)| = 1$ ; this is automatic here due to our stronger hypotheses. The conclusion now follows from the area formula [4, Theorem 1, p. 96].  $\square$

## 6. EXAMPLES OF DENOMINATOR-PRESERVING MAPS

The gingerbreadman map  $F(x, y) = (1 - y + |x|, x)$  is a well known example of a denominator-preserving area-preserving homeomorphism of  $\mathbb{R}^2$  possessing interesting dynamical properties [2]. It has a unique elliptic fixed point at  $(1, 1)$ , which is surrounded by infinitely many polygonal annuli on which the dynamics is hyperbolic and chaotic in regions of positive measure.

Denominator-preserving volume-preserving homeomorphisms of the unit even-dimensional cube are constructed in [11]. These are linked twist maps, fixing the boundary of the cube, ergodic and uniformly hyperbolic throughout the whole domain.

We close this paper presenting two examples of denominator-preserving area-preserving homeomorphisms of  $\mathbb{R}^2$  which are related to (non)differentiability issues. Both of them are defined with the help of an auxiliary function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Given such an  $f$  we define  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $F(x, y) = (x, y + f(x))$ ;  $F$  is then an area-preserving bijection, is a homeomorphism iff  $f$  is continuous, and is differentiable at every point of the line  $\{x = \alpha\}$  iff  $f$  is differentiable at  $\alpha$ . Moreover,  $F$  preserves all denominators, provided that  $f(l) \in (\text{den}(l))^{-1}\mathbb{Z}$  for every  $l \in \mathbb{Q}$ . Indeed, say that  $u = d^{-1}(a, b)$  has denominator  $d$ . Then  $l = a/d$  has denominator  $e = d/(a, d)$  and  $f(l) = (a, d)c/d$  for some  $c \in \mathbb{Z}$ . Hence  $F(u) = d^{-1}(a, b + (a, d)c)$  has denominator  $d$ , since  $(a, b, d) = 1$  implies  $(a, b + (a, d)c, d) = 1$ . The same argument applies to  $F^{-1}$  (which is induced by  $-f$ ), and hence both  $F$  and  $F^{-1}$  map points of a certain denominator to points of the same denominator, so both of them preserve all denominators.

We construct our first example by defining a sequence  $p_1, p_2, \dots$  of prime numbers as follows:  $p_1$  is 2, and  $p_{t+1}$  is the least prime strictly greater than  $(1 + 1/t)p_t$ . The sequence  $t/p_t$  is strictly decreasing (the initial terms are  $1/2, 2/5, 3/11, 4/17, 5/23, \dots$ ), and converges to 0 because it is dominated by the sequence  $t/(\text{the } t\text{-th prime number})$ , that converges to 0 by the Prime Number Theorem. For each  $t$  choose two rational numbers  $a_t/b_t, c_t/d_t$  such that

$$\frac{c_{t+1}}{d_{t+1}} \leq \frac{a_t}{b_t} < \frac{t}{p_t} < \frac{c_t}{d_t},$$

and each of the two intervals  $[a_t/b_t, t/p_t], [t/p_t, c_t/d_t]$  is unimodular, i.e.,

$$\begin{vmatrix} t & a_t \\ p_t & b_t \end{vmatrix} = \begin{vmatrix} c_t & t \\ d_t & p_t \end{vmatrix} = 1.$$

This is easily accomplished either by using continued fractions or by using Farey partitions [5, Chapter III]. Define now  $f$  by

$$f(x) = \begin{cases} b_t x - a_t, & \text{if } a_t/b_t \leq x \leq t/p_t; \\ -d_t x + c_t, & \text{if } t/p_t \leq x \leq c_t/d_t; \\ 0, & \text{otherwise.} \end{cases}$$

Then  $f$  is continuous and satisfies  $f(l) \in (\text{den}(l))^{-1}\mathbb{Z}$  for every  $l \in \mathbb{Q}$ . Moreover, it is differentiable at each  $\alpha \in \mathbb{R} \setminus \bigcup_{t \geq 1} \{a_t/b_t, t/p_t, c_t/d_t\}$ . Indeed, the only problematic point is  $\alpha = 0$ . However, the ratio  $f(x)/x$  is 0 for  $x$  outside the intervals

$[a_t/b_t, c_t/d_t]$ , and has value

$$M_t(x) = \frac{\min\{b_t x - a_t, -d_t x + c_t\}}{x},$$

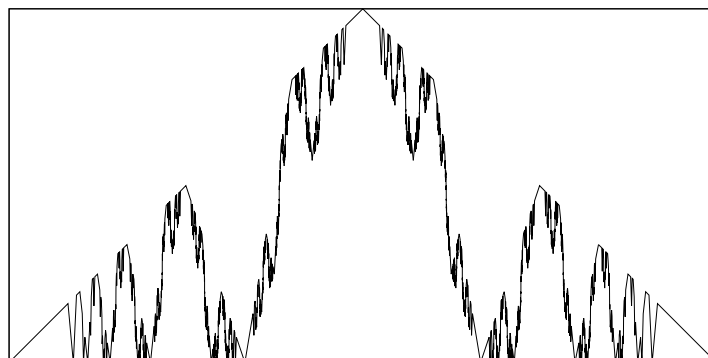
on  $[a_t/b_t, c_t/d_t]$ . It is easily checked that  $M_t$  attains its maximum value  $(1/p_t)/(t/p_t) = 1/t$  at  $x = t/p_t$ ; hence  $f$  is differentiable at 0 with derivative 0. The map  $F$  induced by  $f$  as above is then a homeomorphism of  $\mathbb{R}^2$  which preserves all denominators, is differentiable at the origin with differential the identity map—in accordance with Theorem 5.2—but is not affine in any neighborhood of the origin.

In our second example we construct a nowhere differentiable denominator-preserving homeomorphism of  $\mathbb{R}^2$ . Recall the construction of the Stern-Brocot sequence on the real unit interval [6]; the only interval *belonging* to stage 0 is  $[0/1, 1/1]$ . At stage  $t + 1$  each of the  $2^t$  intervals  $[a/b, c/d]$  belonging to stage  $t$  is split into two intervals  $[a/b, (a+c)/(b+d)]$  and  $[(a+c)/(b+d), c/d]$ ; the point  $(a+c)/(b+d)$  is the *Farey mediant* of  $a/b$  and  $c/d$ . For  $t = 1, 2, 3, \dots$ , define  $g_t : [0, 1] \rightarrow \mathbb{R}$  as follows:

- $g_t = 0$  at all endpoints of all intervals belonging to stage  $t - 1$ ;
- $g_t = 1/\text{den}(u)$  at each Farey mediant  $u$  appearing at stage  $t$ ;
- $g_t$  is affine linear on each interval belonging to stage  $t$ .

The functions  $g_t$  are sawlike,  $g_1 \geq g_2 \geq g_3 \geq \dots$  on  $[0, 1]$ ,  $\|g_t\|_\infty = 1/(t+1)$ , each interval  $[u, v]$  belonging to stage  $t$  is unimodular, and  $g_t$  has slope  $\pm \text{den}(u) \in \mathbb{Z} \setminus \{0\}$  on  $[u, v]$ , provided that the Farey mediant inserted at step  $t$  is  $v$ ; otherwise  $g_t$  has slope  $\pm \text{den}(v)$  on  $[u, v]$ . Consider the alternating sum  $\sum_{1 \leq t} (-1)^{t-1} g_t$ ; by the Leibniz test it converges uniformly to a continuous function  $\bar{f}$ . Given a rational number  $0 \leq l \leq 1$ , we have  $f(l) \in (\text{den}(l))^{-1} \mathbb{Z}$ ; indeed  $l$  appears as an endpoint at some stage  $t$  of the procedure, and then  $g_m(l) = 0$  for every  $m > t$ . Hence  $f(l) = \sum_{1 \leq m \leq t} (-1)^{m-1} g_m(l)$ , and each  $g_m(l)$  is in  $(\text{den}(l))^{-1} \mathbb{Z}$ .

We extend  $f$  to a period-1 function defined on all of  $\mathbb{R}$  in the obvious way, and we claim that  $f$  is of Weierstrass type, i.e., is nowhere differentiable. Indeed, let  $0 \leq \alpha \leq 1$ , and let  $I_1 \supset I_2 \supset I_3 \supset \dots$  be a chain of intervals with  $I_t$  belonging to stage  $t$  and  $\bigcap_{t \geq 1} I_t = \{\alpha\}$  (this chain is unique iff  $\alpha \neq 0, 1$  is irrational, and is determined by the continued fraction expansion of  $\alpha$ ). Let  $I_t = [u_t, v_t]$ , let  $r_t$  be the slope of  $g_t$  on  $I_t$ , and let  $s_t = (f(v_t) - f(u_t))/(v_t - u_t)$ . Since each  $g_m$ , for  $1 \leq m \leq t$ , is linear on  $I_t$ , and  $g_m(u_t) = g_m(v_t) = 0$  for  $m > t$ , the number  $s_t$  is precisely the slope of  $f_t = \sum_{1 \leq m \leq t} (-1)^{m-1} g_m$  on  $I_t$ , i.e.,  $s_t = \sum_{1 \leq m \leq t} (-1)^{m-1} r_m$ . If  $f$  were differentiable at  $\alpha$  then, by Lemma 5.3,  $\lim_{t \rightarrow \infty} s_t$  would exist. But this is impossible, since  $r_m \in \mathbb{Z} \setminus \{0\}$  for each  $m$ .

Graph of  $f_{11}$ .

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